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Onsager Symmetry from Microscopic *TP* Invariance

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We prove for a class of conservative interacting particle systems the validity of Onsager reciprocity relations for the transport coefficients if the microscopic dynamics is invariant under the combined action of time reversal and parity. The situation is reminiscent of *TCP* invariance in quantum field theory. Our systems include multicomponent zero-range and exchange dynamics.

KEY WORDS: Onsager symmetry; microscopic nonreversibility; *TP* invariance; interacting particle systems.

1. INTRODUCTION

Even if the evolution of thermodynamic systems is irreversible, one can discover traces of the time reversibility of the underlying microscopic system. The most well known consequences of microreversibility are perhaps Onsager's symmetry of the transport coefficients[(1)] and the Onsager–Machlup time reversal relation for the most probable trajectories creating a fluctuation or relaxing it to equilibrium.[(2)] The question naturally arises whether these macroscopic consequences of microreversibility, that we shall denote briefly as macroscopic reversibility, can be obtained under more general conditions for the microscopic dynamics. This possibility is not usually discussed in the physical literature, in particular in textbooks, where the prevailing attitude is well summarized by the following statement from the introduction of a recent treatise by R. L. Stratonovich[(3)]: "If the principle of time reversibility is invalid, then the number of useful relations

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of nonequilibrium thermodynamics is reduced drastically, and so the theory becomes much poorer."

Actually it is trivial to see that Onsager symmetry does not require microscopic reversibility as a necessary condition. Consider in fact n uncoupled one component thermodynamic systems some of which are microscopically irreversible. The fact that they are uncoupled means that the corresponding hydrodynamic equations are uncoupled. Assuming that these equations are pure diffusions this means that the diffusion matrix is diagonal. In this case the Onsager matrix is also diagonal and therefore is symmetric.

A less trivial situation will be obtained if we can construct models in which at least one of these matrices is not diagonal. In a recent paper[(4)] we constructed a microscopically non reversible lattice gas with zero range interaction which provides a very simple example of this situation. By microscopic reversibility we understand the validity of the detailed balance condition, that is, the self-adjointness of the generator of the dynamics with respect to the equilibrium measure.

One of the results of the present paper is that the model[(4)] belongs to a rather wide class, which includes also multicomponent exclusion systems, sharing the same property. The reason why this class of models satisfies Onsager symmetry has to be found in the special structure of the generators of the dynamics. In fact, one discovers that in spite of the fact that they are not self-adjoint with respect to the equilibrium measure, these generators are connected to their adjoints by a a parity operation and their invariant measures are invariant under this operation. The proof generalizes that given by Eyink, Lebowitz, and Spohn[(5)] for the case of microscopic reversibility.

The class of systems for which we have proved Onsager symmetry is restricted by the fact that a rigorous derivation of hydrodynamics is possible only in special cases. The class of non reversible models considered in this paper give rise at the macroscopic level to nonlinear purely diffusive equations in the terminology of ref. 10. The equations are of the following form

$$\partial_t \boldsymbol{\rho} = \sum_{i,\,j=1}^{d} \partial_{u_i} \{ D_{i,\,j}(\boldsymbol{\rho}) \cdot \partial_{u_j} \boldsymbol{\rho} \}$$

where $\boldsymbol{\rho}(u, t) = (\rho_1(u, t),..., \rho_n(u, t))$ is a vector standing for the densities of different kinds of particles and $D_{i,\,j}$ is in general a non symmetric $n \times n$ matrix.

Associated to our models there is an entropy functional $S(\boldsymbol{\rho})$ that is written as the integral of a density $s(\boldsymbol{\rho})$: $S(\boldsymbol{\rho}) = \int s(\boldsymbol{\rho}(u))\, du$.

The Onsager coefficients are defined in this context by

$$L(\boldsymbol{\rho}) = D(\boldsymbol{\rho}) \cdot R(\boldsymbol{\rho})$$

where the matrix R is determined by the entropy density $s(\boldsymbol{\rho}(u))$ in the following way

$$(R^{-1})_{a,b}\,(\boldsymbol{\rho}(u)) = \frac{\partial^2}{\partial \rho_a(u)\,\partial \rho_b(u)}\,s(\boldsymbol{\rho}(u)) \tag{1.1}$$

which is by definition a symmetric matrix. Onsager's reciprocity relations mean that L is a symmetric matrix, a property which holds for our models.

We believe that our result should extend to a wider class of translational invariant systems admitting a hydrodynamic description and invariant microscopically under TP. The special role of parity is presumably connected to translational invariance.

The results of the present work remind of quantum field theory, in particular of the TCP invariance. As it is well known in field theory the vacuum state, which in our analogy corresponds to the invariant measure, is invariant under the application of each of these operations T, C, P separately but there are relationships among scattering or reaction amplitudes which do not require the invariance of the dynamics under each operation.

Physicists have generalized Onsager symmetry to stationary non equilibrium situations.[6] In a forthcoming paper we shall discuss also this problem within a rigorous treatment of interacting particle models.

All the models considered above are conservative. In a paper of the authors in collaboration with M. E. Vares,[7] we have analyzed a class of dissipative systems whose dynamics is a superposition of the Kawasaki and Glauber dynamics.[8, 9] These are one component models and we were able to show that the Onsager–Machlup time reversal relation can still hold in absence of microscopic reversibility.

2. MULTI COMPONENT ZERO RANGE PROCESSES

For simplicity, we shall restrict ourselves to one dimensional two component models but all analysis can be carried out for any space dimension and for any number of components. Moreover, our results are not restricted to the neighborhood of the equilibrium.

We consider an interacting particle system that describes the evolution of two types of particles on the discrete one dimensional torus with N points, denoted by $\mathbb{T}_N$ (the integers modulo N). Sites of $\mathbb{T}_N$ are denoted by

x, y and the configurations by the Greek letter $\boldsymbol{\eta}=(\eta_1, \eta_2)$ so that $\eta_a(x)$ stands for the total number of particles of type a at site x for the configuration $\boldsymbol{\eta}$. The stochastic dynamics can be described as follows. Fix a non-negative function $g\colon \mathbb{N} \to \mathbb{R}_+$ that stands for the rate at which particles jump, and two finite range, mean zero transition probability $p_a(\cdot)$, $a=1, 2$, on $\mathbb{Z}$ (for $a=1, 2$, $\sum_y y p_a(y)=0$ and $p_a(x)=0$ for $|x|$ large enough). We shall assume the jump rate g to vanish at 0, to be Lipschitz and to diverge at infinity: $g(0)=0<g(i)$ for $i\geqslant 1$, $|g(k+1)-g(k)|\leqslant l_0$ and $\lim_{k\to\infty} g(k)=\infty$. If there are k_a, $a=1, 2$, particles of type a at a site x of $\mathbb{Z}$, at rate $p_a(y)\, g(k_1+k_2)\{k_a/k_1+k_2\}$ one particle of type a jumps from site x to $x+y$. This happens independently at each site.

The generator Ω_N of this Markov process acts on functions f as

$$\Omega_N f=\frac{N^2}{2}\sum_{a=1}^{2}\sum_{x,\, y\in \mathbb{T}_N} p_a(y)\, T_a^{x,\, x+y} f$$

where the addition in $\mathbb{T}_N$ means addition modulo N; the operators $\{T_1^{x,\, y}, x, y\in \mathbb{T}_N\}$ are defined by

$$(T_1^{x,\, y} f)(\eta_1, \eta_2)=r_x(\boldsymbol{\eta})\, \eta_1(x)[f(\eta_1^{x,\, y}, \eta_2)-f(\eta_1, \eta_2)]$$

with $r_x(\boldsymbol{\eta})=g(\eta_1(x)+\eta_2(x))/\{\eta_1(x)+\eta_2(x)\}$ and $\zeta^{x,\, y}$ is the configuration obtained from ζ letting one particle jump from x to y:

$$\zeta^{x,\, y}(z)=\begin{cases}\zeta(z) & \text{if } z\neq x, y\\ \zeta(z)-1 & \text{if } z=x\\ \zeta(z)+1 & \text{if } z=y\end{cases}$$

The operators $\{T_2^{x,\, y}, x, y\in \mathbb{T}_N\}$ are defined in a similar way.

This process has two conserved quantities: the total number of η_1-particles and the total number of η_2-particles. It is therefore expected that for each fixed density $\rho_a\geqslant 0$ there should exist an equilibrium state with global density of η_a-particles equal to ρ_a.

To describe these equilibrium probability measures, for each $\varphi_1, \varphi_2\geqslant 0$, consider the product probability measure $\nu^N_{\varphi_1,\, \varphi_2}$ on $\mathbb{N}^{\mathbb{T}_N}\times \mathbb{N}^{\mathbb{T}_N}$ defined by

$$\begin{aligned}\nu^N_{\varphi_1,\, \varphi_2}&\{(\eta, \xi);\, \eta(x)=k_1, \xi(x)=k_2\}\\ &=\frac{1}{Z(\varphi_1, \varphi_2)}\,\frac{\varphi_1^{k_1}\varphi_2^{k_2}}{g(k_1+k_2)!}\,\frac{(k_1+k_2)!}{k_1!\, k_2!}\end{aligned}\tag{2.2}$$

for $k_1 \geqslant 0$ and $k_2 \geqslant 0$. In this formula $Z(\varphi_1, \varphi_2)$ is a normalizing constant and $g(k)!$ stands for $g(1) \cdots g(k)$.

Denote by $E_{\nu^N_{\varphi_1, \varphi_2}}[\,\cdot\,]$ expectation with respect to the measure $\nu^N_{\varphi_1, \varphi_2}$ and by $\langle \cdot, \cdot \rangle_{\varphi_1, \varphi_2}$ the inner product in $L^2(\nu^N_{\varphi_1, \varphi_2})$. A simple computation shows that these measures are invariant for the Markov process with generator Ω_N. They are reversible, that is, the principle of detailed balance holds, if the generator is also selfadjoint with respect to these measures, i.e., if $\langle f, \Omega_N g \rangle_{\varphi_1, \varphi_2} = \langle \Omega_N f, g \rangle_{\varphi_1, \varphi_2}$ for every f, g in $L^2(\nu^N_{\varphi_1, \varphi_2})$. This is possible if and only if the transition probabilities $p_a(\cdot)$ are even functions for $a = 1, 2$. If this is not the case, denote by Ω^*_N the adjoint of Ω_N in $L^2(\nu^N_{\varphi_1, \varphi_2})$. A simple computation shows that Ω^*_N corresponds to the generator of a two–component zero range process with the same jump rate g and transition probability $p^*_a(\cdot)$ given by $p^*_a(y) = p_a(-y)$, $a = 1, 2$. The probability measures $\nu^N_{\varphi_1, \varphi_2}$ are invariant for Ω^*_N and Ω_N, Ω^*_N give rise to the same hydrodynamics. Furthermore, we have the following relationship between Ω_N and Ω^*_N

$$\Omega^*_N = P \Omega_N P$$

where P: $\mathbb{N}^{\mathbb{Z}^d} \times \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{N}^{\mathbb{Z}^d} \times \mathbb{N}^{\mathbb{Z}^d}$ is the parity operator defined as follows

$$[P\boldsymbol{\eta}](x) \equiv \boldsymbol{\eta}(-x), \qquad x \in \mathbb{Z}$$

and the action on functions is defined in the usual way:

$$[Pf](\boldsymbol{\eta}) \equiv f(P\boldsymbol{\eta})$$

Notice that the invariant measures $\nu^N_{\varphi_1, \varphi_2}$ are invariant under P. Since Ω_N and Ω^*_N give rise to the same hydrodynamics, they are associated to the same Onsager matrix L. On the other hand, the Onsager matrices corresponding to Ω_N and Ω^*_N must be one the transposed of the other which implies that in this case they must be symmetric. This is confirmed by the explicit calculation. The hydrodynamic equations can be easily written down for the present situation which is a generalization of the models discussed in ref. 4.

Define ρ_a: $\mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by $\rho_a(\varphi_1, \varphi_2) = E_{\nu^N_{\varphi_1, \varphi_2}}[\eta_a(0)]$ and set $\hat{\rho} = \rho_1 + \rho_2$. One can check that $\hat{\rho}$ is a smooth strictly increasing function of $\varphi_1 + \varphi_2$. Denote by $a = a(\hat{\rho})$ the inverse of $\hat{\rho}$: $a(\cdot) = (\hat{\rho})^{-1}(\cdot)$. A simple computation, relying on the explicit formula (2.2) of the invariant measure, shows that $\varphi_a = (\rho_a / \rho_1 + \rho_2)\, a(\rho_1 + \rho_2)$. In conclusion, for each fixed density (ρ_1, ρ_2) we obtained an invariant state with total density of η_a-particles equal to ρ_a. To keep notation simple we shall denote by $b(\hat{\rho})$ the function $a(\hat{\rho})/\hat{\rho}$. We shall from now on fix a density $\bar{\boldsymbol{\rho}} = (\bar{\rho}_1, \bar{\rho}_2)$.

Let us consider now the unit interval $\mathbb{T}=[0,1)$ with periodic boundary conditions and functions $\gamma_a\colon \mathbb{T}\to\mathbb{R}_+$, $a=1,2$ with global density $\bar{\rho}_a$: $\int_{\mathbb{T}} \gamma_a(u)\,du=\bar{\rho}_a$. The main object of our study is the empirical measure $\boldsymbol{\pi}^N(t)=(\pi_1^N(t),\pi_2^N(t))$ defined by:

$$\pi_a^N(t)=\frac{1}{N}\sum_{y\in\mathbb{T}_N}\eta_a(t,y)\,\delta_{y/N}$$

where, for u in $\mathbb{T}$, δ_u is the Dirac measure concentrated at u and $\eta_a(t,y)$ the number of a-particles at site y at time t. If we denote by $Q^N_{\gamma_1,\gamma_2}$ the distribution law of the trajectories $\boldsymbol{\pi}^N(t)$ when the initial measure is concentrated on a configuration pair (η_1^N,η_2^N) such that $\boldsymbol{\pi}^N(0)\to(\gamma_1(u)\,du, \gamma_2(u)\,du)$, as $N\uparrow\infty$, it is possible to show that $Q^N_{\gamma_1,\gamma_2}$ converges weakly, as $N\uparrow\infty$, to the measure concentrated on the path $\boldsymbol{\rho}(t,u)\,du$ whose density is the unique solution of

$$\begin{cases}\partial_t\boldsymbol{\rho}=(1/2)\,\partial_u\{D(\boldsymbol{\rho})\cdot\partial_u\boldsymbol{\rho}\}\\ \boldsymbol{\rho}(0,\cdot)=\boldsymbol{\gamma}(\cdot)\end{cases}\tag{2.3}$$

In this formula $D=D(\boldsymbol{\rho})$ is the nonsymmetric diffusion matrix given by

$$D(\boldsymbol{\rho})=b(\hat{\rho})\,\Sigma+b'(\hat{\rho})\,J(\boldsymbol{\rho})\tag{2.4}$$

where Σ is the two by two diagonal matrix with entries $\alpha_{a,b}$ equal to $\sigma_a^2\delta_{a,b}$ and $J(\boldsymbol{\rho})$ is the matrix with entries $J_{a,b}(\boldsymbol{\rho})=\sigma_a^2\rho_a$. In these formulas, σ_a^2 stand for the variance of the transition probability $p_a(\cdot)$ for $a=1,2$: $\sigma_a^2=\sum_y y^2p_a(y)$.

The above result is a law of large numbers that shows that the empirical measure, in the limit of large N, behaves deterministically according to Eq. (2.3). This result can be proved through the entropy method introduced by Guo, Papanicolaou, and Varadhan in ref. 13. We refer to refs. 11 and 12 for a detailed proof in the case of a one component system.

To compute the Onsager matrix we need the entropy functional which is obtained from the invariant measure:

$$S(\boldsymbol{\gamma})=\int_{\mathbb{T}} s(\boldsymbol{\gamma}(u))\,du$$

where

$$s(\boldsymbol{\gamma})=\sum_{a=1}^{2}E(\gamma_a(u))+F(\gamma_1(u)+\gamma_2(u))$$

is the entropy density and $E(\alpha)=\int_1^\alpha \log \alpha' \, d\alpha'$, $F(\alpha)=\int_1^\alpha \log b(\alpha') \, d\alpha'$. A simple computation shows that the matrix R defined by Eq. (1.1) is such that

$$(R^{-1})_{a,b}(\boldsymbol{\gamma})=\delta_{a,b}\,\frac{1}{\gamma_a(u)}+\frac{b'(\hat{\gamma}(u))}{b(\hat{\gamma}(u))}$$

where $\delta_{a,b}$ stands for the delta of Kronecker. The product $L=DR$ can now be computed using the explicit formula for D given in (2.4). The Onsager matrix L in this context is *diagonal* with entries $L_{a,b}$ equal to

$$L_{a,b}(\boldsymbol{\gamma})=(1/2)\,\sigma_a^2\gamma_a b(\hat{\gamma})\,\delta_{a,b}$$

For the present models it is also possible to prove the Onsager–Machlup time reversal relation. For this purpose we ask what is the probability that our system follows a trajectory different from the solution of (2.3) when N is large but not infinite. This probability is exponentially small in N and can be estimated using the methods of the theory of large deviations introduced for the systems of interest in refs. 14, 15, and 9. The main idea consists in introducing a modified system for which the trajectory of interest (fluctuation) is typical being a solution of the corresponding hydrodynamic equation, and then comparing the two evolutions. For this purpose, for each pair of smooth functions $H_a=H_a(t,u)$, $a=1,2$, we consider the time inhomogeneous Markov process defined by the generator

$$\Omega^H_{N,t} f=\frac{N^2}{2}\sum_{a=1}^{2} e^{H_a(t,(y+x)/N)-H_a(t,x/N)}p_a(y)\,T_a^{x,x+y}f$$

with $p(\cdot)$ and $T_a^{x,x+y}$ as previously defined. The function $\mathbf{H}$ can be interpreted as an external field.

The deterministic equation satisfied by the density of the empirical measure is now

$$\begin{cases}\partial_t\boldsymbol{\rho}=(1/2)\,\partial_u\{D(\boldsymbol{\rho})\cdot\partial_u\boldsymbol{\rho}\}-\partial_u\{b(\hat{\rho})\,\mathbf{A}(\boldsymbol{\rho},\mathbf{H})\}\\ \boldsymbol{\rho}(0,\cdot)=\boldsymbol{\gamma}(\cdot)\end{cases}$$

where $\mathbf{A}(\boldsymbol{\rho},\mathbf{H})$ is the vector with components $A_a=\sigma_a^2\rho_a\,\partial_u H_a$.

Given a function $\boldsymbol{\rho}(t,u)$ twice differentiable with respect to u and once with respect to t and such that $\int_{\mathbb{T}}\rho_a(t,u)\,du=\bar{\rho}_a$ this equation determines uniquely up to an additive constant the field $\mathbf{H}=(H_1,H_2)$.

The probability that the original system follows a trajectory different from a solution of (2.3) can now be expressed in terms of the field $\mathbf{H}$. We introduce the large deviation functional

$$I_{0,t_0}(\boldsymbol{\rho}) = \sum_{a=1}^{2} (\sigma_a^2/2) \int_0^{t_0} dt \int_{\mathbb{T}} du\, b(\hat{\rho})\, \rho_a (\partial_u H_a)^2$$

Let $\mathscr{G}$ be a set of trajectories in the interval $[0, t_0]$. The large fluctuation estimate asserts that

$$Q^N_{\gamma_1,\gamma_2}(\mathscr{G}) \simeq e^{-N I_{0,t_0}(\mathscr{G})} \tag{2.5}$$

where

$$I_{0,t_0}(\mathscr{G}) = \inf_{\boldsymbol{\rho} \in \mathscr{G}} I_{0,t_0}(\boldsymbol{\rho}) \tag{2.6}$$

The sign $\simeq$ has to be interpreted as asymptotic equality of the logarithms.

From the Eqs. (2.5), (2.6), one sees that to find the most probable trajectory that connects the equilibrium $\bar{\boldsymbol{\rho}}$ to a certain state $\boldsymbol{\gamma}(u)$ one has to find the trajectory $\boldsymbol{\rho}(t,u)$ that minimizes $I_{-\infty,0}(\boldsymbol{\rho})$ in the set $\mathscr{G}_{\boldsymbol{\gamma}}$ of all trajectories satisfying the boundary conditions

$$\lim_{t \to -\infty} \boldsymbol{\rho}(t,u) = \bar{\boldsymbol{\rho}}, \qquad \boldsymbol{\rho}(0,u) = \boldsymbol{\gamma}(u)$$

It is now possible to prove, following the same approach of ref. 7, that the unique solution of our variational problem is the function $\boldsymbol{\rho}^*(t,u)$ defined by

$$\boldsymbol{\rho}^*(t,u) = \boldsymbol{\rho}(-t,u) \tag{2.7}$$

where $\boldsymbol{\rho}(t,u)$ is the solution of the hydrodynamic equation which relaxes to equilibrium with initial state $\boldsymbol{\gamma}$. $\boldsymbol{\rho}^*(t,u)$ is therefore a solution of the hydrodynamic equation with inverted drift

$$\partial_t \boldsymbol{\rho} = -(1/2)\, \partial_u \{ D(\boldsymbol{\rho}) \cdot \partial_u \boldsymbol{\rho} \}$$

Equation (2.7) is the Onsager–Machlup time-reversal relation.

All the analysis carried out in this section extends to multi component zero range processes evolving on the lattice $\mathbb{Z}^d$.

3. MULTI COMPONENT EXCHANGE DYNAMICS

We introduce in this section a class of multi component exchange models. The stochastic evolution can be described as follows. Two type of particles evolve on the lattice $\mathbb{Z}^d$. The sites of the lattice are denoted by the last letters of the alphabet: x, y, z and the configurations of the state space $\{0, 1, 2\}^{\mathbb{Z}^d}$ are denoted by the Greek letter η. In this way $\eta(x) = 0$ if there is no particle at site x for the configuration η and $\eta(x) = 1, 2$ if there is a particle of type 1, 2. To keep notation simple we denote hereafter by $\boldsymbol{\eta} = (\eta^1, \eta^2)$ the configuration of particles of type 1 and 2. Thus η^1 and η^2 are configurations of $\{0, 1\}^{\mathbb{Z}^d}$ such that $\eta^1(x) = 1$ (resp. $\eta^2(x) = 1$) if and only if $\eta(x) = 1$ (resp. $\eta(x) = 2$).

The dynamics can be informally described as follows: Fix a class of cylinder functions $c_{x,y}$: $\{0, 1, 2\}^{\mathbb{Z}^d} \to \mathbf{R}_+$. We shall often write $c(x, y, \eta)$ for $c_{x,y}(\eta)$. At rate $c(x, y, \eta)$ the occupation variables $\eta(x)$ and $\eta(y)$ are exchanged if the configuration is η. The generator Ω of this process on the lattice $\mathbb{Z}^d$ is thus

$$(\Omega f)(\eta) = \sum_{x,\, y \in \mathbb{Z}^d} c(x, y, \eta)[f(\sigma^{x,y}\eta) - f(\eta)]$$

where $\sigma^{x,y}\eta$ stands for the configuration η with the occupation variables $\eta(x)$ and $\eta(y)$ exchanged:

$$(\sigma^{x,y}\eta)(z) = \begin{cases} \eta(z) & \text{if} \quad z \neq x, y \\ \eta(y) & \text{if} \quad z = x \\ \eta(x) & \text{if} \quad z = y \end{cases}$$

The operators $\sigma^{x,y}$ naturally extend to cylinder functions: $(\sigma^{x,y}f)(\eta) = f(\sigma^{x,y}\eta)$. Denote by $\{S_t, t \geqslant 0\}$ the semigroup associated to the Markov process with generator Ω.

We now introduce the assumptions made on the jump rates $\{c_{x,y}, x, y \in \mathbb{Z}^d\}$. We shall assume that these jump rates are translation invariant and of finite range in the sense that

$$c(x, y, \eta) = c(x - z, y - z, \tau_z \eta)$$

for all z in $\mathbb{Z}^d$ and that $c(x, y, \eta) = 0$ whenever $|x - y|$ is large enough. Here $\tau_x \eta$ stands for the translation of the configuration η by x units, so that $(\tau_x \eta)(z) = \eta(x + z)$ for all x, z in $\mathbb{Z}^d$. Moreover, to avoid singularities we assume that the jump rates $c(x, y, \eta)$ are not degenerated: $c(x, y, \eta) > 0$ if $\eta(x) \neq \eta(y)$ and $|x - y| = 1$.

Fix a positive integer N, denote by Λ_N the cube of linear size $2N+1$ centered at the origin: $\Lambda_N=\{-N,..., N\}^d$ and denote by Ω_N the generator of the Markov process described above restricted to Λ_N with periodic boundary condition. Ω_N is given by

$$(\Omega_N f)(\eta)=\sum_{x,\, y\in\Lambda_N} c(x, y, \eta)[f(\sigma^{x, y}\eta)-f(\eta)] \tag{3.8}$$

Denote by $\{S_t^N, t\geqslant 0\}$ the semigroup associated to Ω_N.

Since we assumed the rates to be non degenerated, there are only two conserved quantities: the total number of particles of type 1 and the total number of particles of type 2. In particular, for each fixed $M_1\geqslant 0$, $M_2\geqslant 0$ such that $M_1+M_2\leqslant|\Lambda_N|$, there exist a unique invariant measure, denoted by $\nu^N_{M_1, M_2}$, concentrated on configurations η with M_a particles of type a, $a=1, 2$. This class of invariant measures inherits the translation invariance property of the dynamics: $E_{\nu^N_{M_1, M_2}}[\tau_x f]=E_{\nu^N_{M_1, M_2}}[f]$ for all cylinder functions f.

We assume that this family of invariant measures has good limit properties as $\Lambda_N\uparrow\mathbb{Z}^d$. More precisely, denote by $\mathscr{A}$ the set $\{(\theta_1, \theta_2), \theta_a\geqslant 0, \theta_1+\theta_2\leqslant 1\}$. We assume that for each $\boldsymbol{\rho}$ in $\mathscr{A}$, as $\Lambda_N\uparrow\mathbb{Z}^d$ and $M_a/|\Lambda_N|\to\rho_a$, the measure $\nu^N_{M_1, M_2}$ converges weakly to a probability measure, denoted by ν_{ρ_1, ρ_2}. Of course this family of probability measure is invariant for the infinite volume dynamics: $E_{\nu_{\rho_1, \rho_2}}[\Omega f]=0$ for every cylinder function f.

Denote by $\mathbb{T}^d$ the d-dimensional torus $(-1, 1]^d$. Fix a profile γ: $\mathbb{T}^d\to\mathscr{A}$ and denote by $\pi^N_a(t)$ the empirical measure obtained assigning mass N^{-d} to each particle:

$$\pi^N_a(t)=N^{-d}\sum_{x\in\mathbb{T}^d_N}\eta^a_t(x)\,\delta_{x/N}$$

$a=1, 2$. Under the previous assumptions on the generator Ω and some additional hypotheses on the invariant measures $\nu_{\boldsymbol{\rho}}$, a law of large numbers for the empirical measure $\pi^N_a(t)$ speeded up by N^2 can be informally derived as in ref. 5. More precisely, one can show that starting from an initial measure concentrated on a configuration pair (η^1, η^2) such that $\boldsymbol{\pi}^N(0)\to(\gamma_1(u)\,du, \gamma_2(u)\,du)$, as $N\uparrow\infty$ the empirical measure diffusively rescaled $\pi^N_a(tN^2)$ converges in probability to an absolutely continuous measure $\pi_a(t)(du)=\rho_a(t, u)\,du$ whose density ρ_a is the solution of a system of parabolic equations of the form

$$\partial_t\boldsymbol{\rho}=\sum_{j, k=1}^{d}\partial_{u_j}\{D_{j, k}(\boldsymbol{\rho})\cdot\partial_{u_k}\boldsymbol{\rho}\}$$

In this formula, $D = \{D_{i,j}^{a,b},\ a, b = 1, 2,\ 1 \leqslant i, j \leqslant d\}$ is the diffusion matrix connected to the Onsager matrix by $L_{i,j} = D_{i,j} R$. Onsager reciprocity relation states that

$$L_{i,j}^{a,b} = L_{j,i}^{b,a} \tag{3.9}$$

for $1 \leqslant i, j \leqslant d$, $a, b = 1, 2$.

The derivation of the hydrodynamic equation can be done in a completely rigorous way for special models. For instance, for the two-colour mean-zero asymmetric simple exclusion process.

As shown in ref. 5 the Onsager matrix L is composed of a static and a dynamic part

$$\begin{aligned} L_{i,j}^{a,b}(\boldsymbol{\rho}) &= -\sum_z z_j \langle \eta^b(z), W_{0,e_i}^a \rangle_{\boldsymbol{\rho}} + \int_0^\infty \sum_z \langle \sigma^{z,z+e_j} S_r W_{0,e_i}^a, W_{z,z+e_j}^b \rangle_{\boldsymbol{\rho}}\, dr \\ &= L_{i,j}^{a,b\ Stat}(\boldsymbol{\rho}) + L_{i,j}^{a,b\ Dyn}(\boldsymbol{\rho}) \end{aligned} \tag{3.10}$$

In this formula $\langle \cdot \rangle_{\boldsymbol{\rho}}$ stands for the expectation with respect to $\nu_{\boldsymbol{\rho}}$ and $W_{z,z+e_j}^a = C_{z,z+e_j}[\eta^a(z) - \eta^a(z+e_j)]$. Onsager relation (3.9) is satisfied if the generator of the process is self-adjoint with respect to the equilibrium measure, i.e., if $\langle \Omega f, g \rangle_{\boldsymbol{\rho}} = \langle \Omega g, f \rangle_{\boldsymbol{\rho}}$ for all f, g in $L^2(\nu_{\boldsymbol{\rho}})$. This means that the stochastic dynamics is reversible.

We want to prove (3.9) under the weaker requirement of *TP* invariance. Let Ω^* denote the adjoint of Ω with respect to $\nu_{\boldsymbol{\rho}}$. It follows from the identity $\langle \Omega f, g \rangle_{\boldsymbol{\rho}} = \langle f, \Omega^* g \rangle_{\boldsymbol{\rho}}$ that Ω^* is given by

$$(\Omega^* f)(\eta) = \sum_{x,y} c^*(x, y, \eta)[f(\sigma^{x,y}\eta) - f(\eta)]$$

where

$$c^*(x, y, \eta) = c(x, y, \sigma^{x,y}\eta) \frac{\nu_{\boldsymbol{\rho}}(\sigma^{x,y}\eta)}{\nu_{\boldsymbol{\rho}}(\eta)} \tag{3.11}$$

We shall assume that the adjoint Ω^* is related to the generator Ω by the parity relation:

$$\Omega^* = P\Omega P$$

where P is the parity operator defined in the previous section: $(P\boldsymbol{\eta})(x) = \boldsymbol{\eta}(-x)$, $(Pf)(\boldsymbol{\eta}) = f(P\boldsymbol{\eta})$.

A simple computation shows that $P\sigma^{x,y}P = \sigma^{-x,-y}$ for x, y in $\mathbb{Z}^d$. In particular, it follows from the identity $\Omega^* = P\Omega P$ that

$$c^*(x, y, \eta) = c(-x, -y, P\eta) \tag{3.12}$$

On the other hand, we claim that $\nu_{\boldsymbol{\rho}}$ is invariant under the parity operation. Fix a cylinder function f and denote by $P\nu_{\boldsymbol{\rho}}$ the probability measure defined by $P\nu_{\boldsymbol{\rho}}(\eta) = \nu_{\boldsymbol{\rho}}(P\eta)$. Changing variables, we have that

$$\int \Omega^* f d(P\nu_{\boldsymbol{\rho}}) = \int P\Omega^* f d\nu_{\boldsymbol{\rho}}$$

Since $\Omega^* = P\Omega P$ and since $\nu_{\boldsymbol{\rho}}$ is invariant for Ω, the previous expression is equal to

$$\int \Omega P f d\nu_{\boldsymbol{\rho}} = 0$$

Therefore $\int \Omega^* f d(P\nu_{\boldsymbol{\rho}}) = 0$ for every cylinder function f. Since Ω and Ω^* have the same invariant measure and $P\nu_{\boldsymbol{\rho}}$ is concentrated on configurations with asymptotic density equal to $\boldsymbol{\rho}$, assuming that there is only one extremal invariant and translation invariant probability measure concentrated on configurations with asymptotic density equal to $\boldsymbol{\rho}$, we have that $P\nu_{\boldsymbol{\rho}} = \nu_{\boldsymbol{\rho}}$.

We are now ready to prove that L satisfies the Onsager reciprocity relations if $\Omega^* = P\Omega P$. We consider separately the static and the dynamic part of the Onsager matrix L. We begin with the static part.

By formula (3.10), $L_{i,j}^{a,b\,\mathrm{Stat}}(\boldsymbol{\rho})$ is equal to

$$-\sum_{x \in \mathbb{Z}^d} x_j \langle \eta^b(x), W^a_{0,e_i} \rangle_{\boldsymbol{\rho}}$$

Fix a cube Λ and denote by $\langle \cdot, \cdot \rangle_{\Lambda, \boldsymbol{\rho}}$ the inner product with respect to $\nu_{\Lambda, \boldsymbol{\rho}}$, the measure $\nu_{\boldsymbol{\rho}}$ restricted to Λ. Assuming that the correlations decay fast enough, the previous sum is equal to

$$-\lim_{\Lambda \uparrow \mathbb{Z}^d} \sum_{x \in \Lambda} x_j \langle \eta^b(x), W^a_{0,e_i} \rangle_{\Lambda, \boldsymbol{\rho}} = -\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{x, y \in \Lambda} x_j \langle \eta^b(x), W^a_{y, y+e_i} \rangle_{\Lambda, \boldsymbol{\rho}}$$

The last equality follows from our assumption on the decay of correlations.

Fix a cube Λ centered at the origin. A simple computation shows that

$$\sum_{x, y \in \Lambda} x_j \langle \eta^b(x), W^a_{y, y+e_i} \rangle_{\Lambda, \boldsymbol{\rho}} = \sum_{x, y \in \Lambda} x_j y_i \langle \eta^b(x), \Omega \eta^a(y) \rangle_{\Lambda, \boldsymbol{\rho}}$$

On the other hand, since Ω^* is the adjoint of Ω and since by assumption $\Omega^* = P\Omega P$, we have that

$$\sum_{x,\, y \in \Lambda} x_j y_i \langle \eta^b(x), \Omega \eta^a(y) \rangle_{\Lambda, \boldsymbol{\rho}} = \sum_{x,\, y \in \Lambda} x_j y_i \langle P\Omega P \eta^b(x), \eta^a(y) \rangle_{\Lambda, \boldsymbol{\rho}}$$

Since $\nu_{\boldsymbol{\rho}}$ is invariant under the parity operation, changing variables in the right hand side of the previous identity, we show that it is equal to

$$\sum_{x,\, y \in \Lambda} x_j y_i \langle \Omega \eta^b(-x), \eta^a(-y) \rangle_{\Lambda, \boldsymbol{\rho}} = \sum_{x,\, y \in \Lambda} x_j y_i \langle \Omega \eta^b(x), \eta^a(y) \rangle_{\Lambda, \boldsymbol{\rho}}$$

This proves that $L_{i,\, j}^{a,\, b\, Stat}(\boldsymbol{\rho}) = L_{j,\, i}^{b,\, a\, Stat}(\boldsymbol{\rho})$.

We turn now to the dynamic part. It follows from formula (3.10) that $L_{i,\, j}^{a,\, b\, Dyn}(\boldsymbol{\rho})$ is given by

$$\int_0^\infty dr \sum_{x \in \mathbb{Z}^d} \langle \sigma^{x,\, x+e_j} S_r W^a_{0,\, e_i}, W^b_{x,\, x+e_j} \rangle_{\boldsymbol{\rho}}$$

Changing variables $\xi = \sigma^{x,\, x+e_j} \eta$, by (3.11) we obtain that the previous expression is equal to

$$\int_0^\infty dr \sum_{x \in \mathbb{Z}^d} \langle S_r W^a_{0,\, e_i}, W^{b,\, *}_{x,\, x+e_j} \rangle_{\boldsymbol{\rho}}$$

In this formula W^* stands for the current with respect to the reversed process. Since by assumption $\Omega^* = P\Omega P$ and $\nu_{\boldsymbol{\rho}}$ is invariant for the parity operator, we have that

$$\langle S_r W^a_{0,\, e_i}, W^{b,\, *}_{x,\, x+e_j} \rangle_{\boldsymbol{\rho}} = \langle P W^a_{0,\, e_i}, S_r P W^{b,\, *}_{x,\, x+e_j} \rangle_{\boldsymbol{\rho}}$$

By (3.12) we have that $PW^a_{x,\, y} = -W^{a,\, *}_{-x,\, -y}$. Changing variables in the summation over x, we obtain that the previous time integral is equal to

$$\int_0^\infty dr \sum_{x \in \mathbb{Z}^d} \langle S_r W^b_{0,\, e_j}, W^{a,\, *}_{x,\, x+e_i} \rangle_{\boldsymbol{\rho}}$$

This concludes the proof of the Onsager reciprocity relations in the case where the adjoint is such that $\Omega^* = P\Omega P$.

We may ask also what happens if in d dimensions the generator and its adjoint are related for example by a reflection of the form $P(x_1,..., x_d) = (-x_1,..., -x_i, x_{i+1},..., x_d)$. It is easy to see that the elements of the Onsager matrix are symmetric if the two space indices belong to the same group (reflected or non reflected) and antisymmetric if the indices belong to different groups.

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